Introduction to Eigenspaces

Recall: Let A be a $n \times n$ matrix. Then a vector $\boldsymbol{x}$ in $\mathbb{R}^n$ is an *eigenvector* of A with corresponding *eigenvalue* λ (a scalar) if and only if

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}, \qquad \boldsymbol{x} \neq \mathbf{0} \tag{1}$$

Definition: Let A be a $n \times n$ matrix and let λ be an eigenvalue of A. The set E_λ defined

$$E_\lambda = \text{null}(A - \lambda I) \tag{2}$$

is called the *eigenspace* of A corresponding to the eigenvalue λ.

Note 1: Since E_λ is the null space of $A - \lambda I$, the eigenspace E_λ is a *subpsace* of $\mathbb{R}^n$.

Note 2: E_λ contains the zero vector and *all* eigenvectors of A with eigenvalue λ.

Example: Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

1. Find the eigenvalues of A.

$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}\right)$

$= (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$

The eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = -1$.

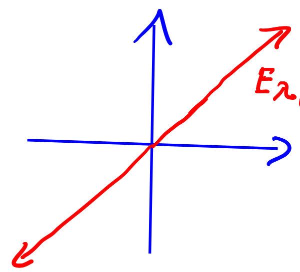
2. One eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is $\lambda_1 = 3$. Find a basis for E_{λ_1} and calculate $\dim(E_{\lambda_1})$. Sketch the eigenspace E_{λ_1}.

$$E_{\lambda_1} = \text{null}(A - \lambda_1 I) = \text{null}(A - 3I) = \text{null}\left(\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\right)$$

$$= \text{null}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 - x_2 = \emptyset \right\}$$

$$= \left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \text{ in } \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \text{ in } \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $\quad \dim(E_{\lambda_1}) = 1$



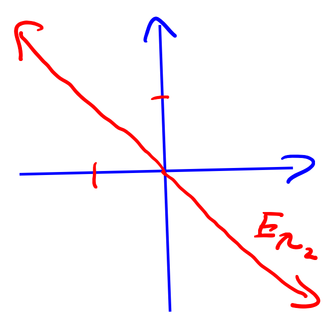
3. The other eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is $\lambda_2 = -1$. Find a basis for E_{λ_2} and calculate $\dim(E_{\lambda_2})$. Sketch the eigenspace E_{λ_2}.

$$E_{\lambda_2} = \text{null}(A - \lambda_2 I) = \text{null}(A + I) = \text{null}\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right)$$

$$= \text{null}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 + x_2 = 0 \right\} = \left\{ t \begin{bmatrix} -1 \\ 1 \end{bmatrix} : t \text{ in } \mathbb{R} \right\}$$

$$= \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right).$$

$\mathcal{B}_2 = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ $\quad \dim(E_{\lambda_2}) = 1$



Example: The matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ has eigenvalue $\underline{\lambda_1 = 1}$. Find a basis for E_{λ_1} and calculate $\dim(E_{\lambda_1})$. Give a geometric description of the eigenspace E_{λ_1}.

$$E_{\lambda_1} = \text{null}(A - \lambda_1 I) = \text{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = \text{null}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + \underset{\underset{t_1}{\uparrow}}{x_2} + \underset{\underset{t_2}{\uparrow}}{x_3} = 0 \right\} = \left\{ \begin{bmatrix} -t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} : t_1, t_2 \text{ in } \mathbb{R} \right\}$$

$$= \left\{ t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : t_1, t_2 \text{ in } \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \dim(E_{\lambda_1}) = 2$$

The eigenspace E_{λ_1} is a $\underline{\text{plane}}$ in $\mathbb{R}^3$.